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J.W. Cohen

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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J.W. Cohen
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT
For the $GI/G/1$-queueing model with traffic load $a<1$, service time distribution $B(t)$ and interarrival time distribution $A(t)$ holds, whenever for $t\to\infty$:

$$1-B(t)\sim\frac{c}{(t/\beta)^{\nu}}+\mathrm{O}(\mathrm{e}^{-\delta t}),\quad c>0,\quad 1<\nu<2,\quad \delta>0,$$

$$\int_0^\infty t^{\mu}\mathrm{d}A(t)<\infty\quad\text{for}\quad \mu>\nu,$$

that $(1-a)^{\frac{1}{\nu-1}}\mathbf{w}$ converges in distribution for $a\uparrow 1$. Here $\mathbf{w}$ is distributed as the stationary waiting time distribution. The L.S.-transform of the limiting distribution is derived and an asymptotic series for its tail probabilities is obtained. The theorem actually proved in the text concerns a slightly more general asymptotic behaviour of $1-B(t)$, $t\to\infty$, than mentioned above.

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Note: work carried out under project LRD

1. Introduction

For the $GI/G/1$ queue denote by $A(t)$ and $B(t)$ the interarrival – and service time distribution and by a the traffic load with $a<1$.

The distribution $B(t)$ is said to have a Pareto type-tail if: for $t\to\infty$,

$$\begin{aligned}&1-B(t)=\frac{c}{(t/\beta)^{\nu}}+\sum_{n=1}^{N}\frac{c_n}{(t/\beta)^{\nu_n}}+\mathrm{O}(\mathrm{e}^{-\delta t}),\\ &1<\nu<2,\quad \beta:=\int_0^\infty t\mathrm{d}B(t)),\quad c>0,\quad \delta>0,\\ &c_n\geq 0,\quad \nu_n>\nu, N \text{ a finite integer } \geq 1.\end{aligned}\tag{1.1}$$

$\mathbf{w}$ shall denote a stochastic variable with distribution $W(t)$, the stationary distribution of the actual waiting time of the $GI/G/1$-model.

Write

$$\Delta:=\left[\frac{1-a}{a}\;\frac{\Gamma(\nu)\sin(v-1)\pi}{c\pi}\right]^{\frac{1}{\nu-1}},\tag{1.2}$$

here $\Gamma(\cdot)$ is the gamma function and x^α, α real, is defined by its principal value, i.e., it is positive for x positive.

THEOREM. *When $B(t)$ has a Pareto-type tail as specified in (1.1) and when*

$$\int_0^\infty t^\mu \mathrm{d}A(t) < \infty \; for \; a \; \mu > \nu, \tag{1.3}$$

then the stochastic variable $(1-a)^{\frac{1}{\nu-1}}\mathbf{w}/\beta$ converges for $a \uparrow 1$ in distribution, and

$$\lim_{a\uparrow 1} \; \mathrm{E}\{\mathrm{e}^{-\rho\Delta\mathbf{w}/\beta}\} = \frac{1}{1+\rho^{\nu-1}}, \quad \mathrm{Re}\,\rho \geq 0; \tag{1.4}$$

the righthand side of (1.4) is the Laplace-Stieltjes transform of a true probability distribution $R_{\nu-1}(t)$ with support $(0,\infty)$; and for $t \to \infty$ and every finite $H \in \{1,2,\dots\}$,

$$1 - R_{\nu-1}(t) = \frac{1}{\pi}\sum_{n=1}^{H}(-1)^{n-1}\frac{\Gamma(n(\nu-1))\sin n(\nu-1)\pi}{t^{n(\nu-1)}} + \mathrm{O}\{t^{-(H+1)(\nu-1)}\}. \tag{1.5}$$

For a special class of Pareto-tailed service time distributions the theorem has been derived for the $M/G/1$-model in [2]. The distribution $R_{\nu-1}(t)$ is called the Kovalenko distribution, cf. [9]. For $\nu = 1\frac{1}{2}$ we have , cf. [1],

$$R_{1/2}(t) = 1 - \frac{2}{\sqrt{\pi}}\mathrm{e}^t\mathrm{Erfc}(t^{1/2}), \quad t > 0, \tag{1.6}$$

with

$$\mathrm{Erfc}(x) = \int_x^\infty \mathrm{e}^{-u^2}\mathrm{d}u.$$

The proof of the theorem is given in the next section, it uses an idea of the proof of Theorem 1, DOETSCH [7], vol. I, p. 467.

The theorem stated above is a heavy traffic result. The classical heavy traffic theorem for the $GI/G/1$-model, cf. [3], Section III.7.2, requires the finiteness of the second moment of $A(t)$ and that of $B(t)$. In a forthcoming paper by O.J. Boxma and the present author generalisations of the theorem above will be discussed.

2. PROOF OF THE THEOREM

We consider first the case with all $c_n = 0$, $n = 1, \dots, N$. Consequently, it is seen from (1.1) that we may write for $t \geq \beta$,

$$1 - B(t) = \frac{c}{(t/\beta)^\nu} + F(t), \tag{2.1}$$

with

$$\int_\beta^\infty \mathrm{e}^{-\rho t}F(t)\mathrm{d}t \text{ convergent for } \mathrm{Re}\,\rho > -\delta, \quad \delta > 0. \tag{2.2}$$

With

$$\beta(\rho) := \int_{0-}^{\infty} \mathrm{e}^{-\rho t} \mathrm{d}B(t), \quad \operatorname{Re}\rho \geq 0, \tag{2.3}$$

we have: for $\operatorname{Re}\rho \geq 0$,

$$\begin{gathered} \frac{1-\beta(\rho)}{\rho\beta} = \int_0^{\infty} \mathrm{e}^{-\rho t}(1-B(t))\frac{\mathrm{d}t}{\beta} = \\ \int_0^{\beta} \mathrm{e}^{-\rho t}\{1-B(t)\}\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} \mathrm{e}^{-\rho t}\frac{c}{(t/\beta)^{\nu}}\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} \mathrm{e}^{-\rho t}F(t)\frac{\mathrm{d}t}{\beta}, \end{gathered} \tag{2.4}$$

and

$$1 = \int_0^{\beta} \{1-B(t)\}\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} \frac{c}{(t/\beta)^{\nu}}\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} F(t)\frac{\mathrm{d}t}{\beta}\cdot$$

It follows: for $\operatorname{Re}\rho \geq 0$,

$$1 - \frac{1-\beta(\rho)}{\rho\beta} = g_1(\rho\beta) - \int_{\beta}^{\infty} \mathrm{e}^{-\rho t}\frac{c}{(t/\beta)^{\nu}}\frac{\mathrm{d}t}{\beta}, \tag{2.5}$$

with

$$g_1(\rho\beta) := \int_0^{\beta} (1-\mathrm{e}^{-\rho t})\frac{1-B(t)}{\beta}\mathrm{d}t + \int_{\beta}^{\infty} \frac{c}{(t/\beta)^{\nu}}\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} \{1-\mathrm{e}^{-\rho t}\}F(t)\frac{\mathrm{d}t}{\beta}\cdot \tag{2.6}$$

By using (2.2) it is readily seen that $g_1(\rho\beta)$ is a regular function of ρ for $\operatorname{Re}\rho > -\delta$.

For the integral in (2.5) we have by partial integration: for $\operatorname{Re}\rho \geq 0$,

$$c\int_{\beta}^{\infty} \mathrm{e}^{-\rho t}\left(\frac{t}{\beta}\right)^{-\nu} \mathrm{d}\frac{t}{\beta} = -g_2(\rho\beta) + c\Gamma(1-\nu)(\rho\beta)^{\nu-1}, \tag{2.7}$$

with

$$g_2(\rho\beta) := \frac{c}{\nu-1}\mathrm{e}^{-\rho\beta} + \frac{c\rho}{1-\nu}\int_0^{\beta} \mathrm{e}^{-\rho t}\left(\frac{t}{\beta}\right)^{1-\nu} \mathrm{d}t. \tag{2.8}$$

Obviously $g_2(\rho\beta)$ is also an entire function of ρ for all ρ, note that $0 < \nu - 1 < 1$.

From, cf. [8], p. 3,

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin\pi\lambda}, \quad \lambda \text{ not an integer}, \tag{2.9}$$

and with

$$g(\rho\beta) := g_1(\rho\beta) + g_2(\rho\beta), \tag{2.10}$$

we have from (2.5), ..., (2.10) for $\mathrm{Re}\,\rho \geq 0$,

$$1-\frac{1-\beta(\rho)}{\rho\beta}=g(\rho\beta)+\frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}(\rho\beta)^{\nu-1}\,. \tag{2.11}$$

From (2.6), (2.8) and (2.10), it is seen that $g(\rho\beta)$ is also a regular function of ρ for $\mathrm{Re}\,\rho > -\delta$. From (2.11) it follows that $g(0)=0$. Hence since $g(\rho\beta)$, $\mathrm{Re}\,\rho > -\delta$ is a regular function we have: for $\mathrm{Re}\,\rho > -\delta$, $|\rho| \to 0$,

$$g(\rho\beta)=\gamma\rho\beta+\mathrm{O}((\rho\beta)^2), \tag{2.12}$$

with γ a finite constant.

Put

$$\alpha(-\rho):=\int_0^\infty \mathrm{e}^{\rho t}\mathrm{d}A(t), \quad \mathrm{Re}\,\rho=0, \tag{2.13}$$

so that $\alpha(-\rho)$ is the characteristic function of the distribution $A(t)$. From (1.1), (1.3) and the series expansion of a characteristic function, cf. [10], p. 199, we have: for $\mathrm{Re}\,\rho=0$, $|\rho|\to 0$,

$$\alpha(-\rho)=1+\alpha\rho+\mathrm{O}(|\rho|^{\mu}), \tag{2.14}$$

$$\alpha:=\int_0^\infty t\mathrm{d}A(t)=\beta/a.$$

Let $\mathbf{i}$ be the idle period, i.e. the difference of a busy cycle and the busy period contained is this busy cycle. The relation between the distributions of $\mathbf{w}$ and $\mathbf{i}$ is given by, cf. [4], p. 21, or [3], p. 371; for $\mathrm{Re}\,\rho=0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}=\frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{-\rho\mathrm{E}\{\mathbf{i}\}}\left[\frac{1-\beta(\rho)\alpha(-\rho)}{(\beta-\alpha)\rho}\right]^{-1}, \tag{2.15}$$

note that

$$\mathrm{E}\{\mathbf{i}\}=(\alpha-\beta)\mathrm{E}\{\mathbf{n}\}, \tag{2.16}$$

with $\mathbf{n}$ the number of customers served in a busy cycle.

With

$$A_\nu:=\frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}\,, \tag{2.17}$$

we have from (2.11): for $\mathrm{Re}\,\rho=0$,

$$\begin{aligned}&\frac{1-\beta(\rho)\alpha(-\rho)}{(\beta-\alpha)\rho}=\\ &\frac{\beta}{\beta-\alpha}\left[\frac{1-\alpha(-\rho)}{\beta\rho}+\{1-g(\rho\beta)\}\alpha(-\rho)-A_\nu(\rho\beta)^{\nu-1}\alpha(-\rho)\right].\end{aligned} \tag{2.18}$$

Put, cf. (1.3),

$$\sigma := \min(1, \mu - 1) > 0. \tag{2.19}$$

By using (2.12) and (2.14) it follows from (2.18) since $\mu > \nu$, cf. (1.3), that for $\mathrm{Re}\,\rho = 0$, $|\rho| \to 0$,

$$\begin{aligned} \frac{1-\beta(\rho)\alpha(-\rho)}{(\beta-\alpha)\rho} &= \frac{\beta}{\beta-\alpha}\left[-\frac{\alpha}{\beta} + 1 - A_\nu(\rho\beta)^{\nu-1} + \mathrm{O}(|\rho|^\sigma)\right] \\ &= 1 + \frac{a}{1-a}\{A_\nu(\rho\beta)^{\nu-1} + \mathrm{O}(|\rho|^\sigma)\}. \end{aligned} \tag{2.20}$$

Put for $\mathrm{Re}\,r \geq 0$, cf. (1.2),

$$\rho = \left[\frac{1-a}{a}A_\nu^{-1}\right]^{\frac{1}{\nu-1}} r/\beta = \Delta r/\beta. \tag{2.21}$$

With for $\mathrm{Re}\,r \leq 0$, $a < 1$,

$$\omega(\rho) := \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}, \qquad \chi(\rho) := \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\}}{\rho\mathrm{E}\{\mathbf{i}\}}. \tag{2.22}$$

we have from (2.15), (2.20) and (2.21): for $0 < 1-a << 1$ and $\mathrm{Re}\,r = 0$,

i. $\omega(r\Delta/\beta) = \chi(-r\Delta/\beta)\left[1 + r^{\nu-1} + r^\sigma\mathrm{O}\left((1-a)\right)^{\frac{1-\nu+\sigma}{\nu-1}})\right]^{-1}$,

ii. $\omega(r\Delta/\beta)$ and $\chi(-r\Delta/\beta)$ are both regular for $\mathrm{Re}\,r > 0$, continuous for $\mathrm{Re}\,r \geq 0$,

iii. $$\begin{array}{lll} |\omega(r\Delta/\beta)| \leq 1, & |\chi(-r\Delta/\beta)| \leq 1, & \mathrm{Re}\,\rho \geq 0, \\ \omega(0) = 1, & \chi(0) = 1. & \end{array}$$

The conditions (2.23)i formulate for $\omega(r\Delta/\beta)$ and $\chi(-r\Delta/\beta)$ a boundary value problem of a type as discussed in [6]. It is not difficult to verify that the conditions (26)i,...,iv of [6] are fulfilled for the present boundary value problem with $0 < 1-a << 1$. Hence from (31) of [6] its solution reads: for $0 < 1-a << 1$,

$$\begin{aligned} \omega(r\Delta/\beta) &= \mathrm{e}^{H(r\Delta/\beta)}, \quad \mathrm{Re}\,r > 0, \\ \chi(-r\Delta/\beta) &= \mathrm{e}^{H(r\Delta/\beta)}, \quad \mathrm{Re}\,r < 0, \end{aligned} \tag{2.23}$$

with

$$H(r\Delta/\beta) := \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\left[1 + \eta^{\nu-1} + \eta^\sigma\mathrm{O}((1-a)^{\frac{1-\nu+\sigma}{\nu-1}})\right]\frac{r\mathrm{d}\eta}{(\eta-r)\eta}.$$

This integral is a principal value, singular Cauchy integral, cf. [5], Section 1.1.5 and [6]. The integral is absolutely convergent and it follows readily by contour integration in the right half plane that

$$\begin{aligned} \lim_{a\uparrow 1} H(r\Delta/\beta) &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\{1+\eta^{\nu-1}\}\frac{r\mathrm{d}\eta}{(\eta-r)\eta} \\ &= -\log\{1+r^{\nu-1}\} \qquad \text{for} \quad \mathrm{Re}\,r \geq 0, \\ &= 0 \qquad\qquad\qquad\quad\ \text{for} \quad \mathrm{Re}\,r \leq 0, \end{aligned} \tag{2.24}$$

note that the logarithm of the integrand is regular for $\mathrm{Re}\,\eta > 0$, continuous for $\mathrm{Re}\,\eta \geq 0$ and zero for $\eta = 0$, cf. further [5], Section 1.1.5. Hence from (2.24) and (2.25): for $\mathrm{Re}\,r \geq 0$,

$$\begin{aligned} \lim_{a\uparrow 1} \omega(r\Delta/\beta) &= \lim_{a\uparrow 1} \mathrm{E}\{e^{-r\Delta\mathbf{w}/\beta}\} = \frac{1}{1+r^{\nu-1}}, \\ \lim_{a\uparrow 1} \chi(r\Delta/\beta) &= 1. \end{aligned} \tag{2.25}$$

By using Feller's continuity theorem for L.S. transforms of probability distributions it follows that $\Delta\mathbf{w}/\beta$ converges in distribution for $a \uparrow 1$, with limiting distribution $R_{\nu-1}(t)$ given by

$$\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}R_{\nu-1}(t) = \frac{1}{1+r^{\nu-1}}, \qquad \mathrm{Re}\,r \geq 0. \tag{2.26}$$

It remains to prove (1.5).

$$\int_0^\infty \mathrm{e}^{-rt}\{1 - R_{\nu-1}(t)\}\mathrm{d}t = \frac{1}{r}\{1 - \frac{1}{1+r^{\nu-1}}\} = \frac{r^{\nu-2}}{1+r^{\nu-1}}\,. \tag{2.27}$$

Because $1 < \nu < 2$, the righthand side can be continued analytically out from $\mathrm{Re}\,r \geq 0$, into $\{r : |\arg r| \leq \psi, \frac{1}{2}\pi < \psi < \pi\}$. With D the contour defined by: for a $r_0 > 0$,

$$D := \{r : r = r_0\mathrm{e}^{\mathrm{i}\phi},\ \phi = \pm\psi\} \cup \{r : r = R\mathrm{e}^{\pm\mathrm{i}\psi}, R \geq r_0\}, \tag{2.28}$$

it is readily shown by starting from the inversion integral for the Laplace transform that

$$1 - R_{\nu-1}(t) = \frac{1}{2\pi i}\int_D \mathrm{e}^{rt}\frac{r^{\nu-2}}{1+r^{\nu-1}}\mathrm{d}r, \tag{2.29}$$

with the direction on D such that on $r = r_0\mathrm{e}^{\mathrm{i}\phi}$ it is counterclockwise with respect to the origin. For $r = |r_0| < 1$ we have

$$\frac{r^{\nu-2}}{1+r^{\nu-1}} = \frac{1}{r}\sum_{n=0}^{\infty}(-1)^{n-1}r^{n(\nu-1)}. \tag{2.30}$$

We now apply a theorem of DOETSCH [7], vol. II, p. 159 to derive an asymptotic series for $1 - R_{\nu-1}(t)$, $t \to \infty$. It is not difficult to show that this theorem may be applied here. It uses the relation

$$\frac{1}{2\pi i}\int_D \mathrm{e}^{rt}r^{\lambda}\mathrm{d}\lambda = \frac{1}{\Gamma(-\lambda)}t^{-\lambda-1}, \quad \lambda \neq 0, 1, 2, \dots,$$

and it states that: for $t \to \infty$ and every finite $H \in \{1, 2, \dots\}$.

$$1 - R_{\nu-1}(t) = \sum_{n=1}^{H}(-1)^{n-1}\frac{t^{-n(\nu-1)}}{\Gamma(1-n(\nu-1))} + \mathrm{O}(t^{-(H+1)(\nu-1)}). \tag{2.31}$$

By using the relation (2.9) the relation (1.5) follows, and the theorem has been proved for the case $c_n = 0$, $n = 1, \dots, N$.

To complete the proof for $c_n > 0$, it suffices to take $c_1 > 0$, $c_2 = \ldots = c_N = 0$, since it is readily seen that the general case proceeds along the same lines. However, we have to distinguish the case that $\nu_1 (> \nu)$ is not an integer and that of ν_1 is an integer ≥ 2.

First, we consider the case $c_1 > 0$, ν_1 noninteger. Instead of (2.1) we write: for $t \geq \beta$,

$$1 - B(t) = \frac{c}{(t/\beta)^{\nu}} + \frac{c_1}{(t/\beta)^{\nu_1}} + F(t), \tag{2.32}$$

with $F(t)$ again satisfying (2.2). By repeated partial integration it is readily shown, cf. [7], vol. II, p. 468, and (2.7), that: for $\mathrm{Re}\, \rho \geq 0$,

$$c_1 \int_{\beta}^{\infty} \mathrm{e}^{-\rho t} \frac{1}{(t/\beta)^{\nu_1}} \frac{\mathrm{d}t}{\beta} = -g_2(\rho\beta) + c_1 \Gamma(1 - \nu_1)(\rho\beta)^{\nu_1 - 1}, \tag{2.33}$$

with $g_2(\rho\beta)$ an entire function of ρ for $\mathrm{Re}\, \rho > -\delta$.

The relation (2.11) is now replaced by: for $\mathrm{Re}\, \rho \geq 0$,

$$\begin{aligned} 1 - \frac{1 - \beta(\rho)}{\beta\rho} = g(\rho\beta) \quad &+ \quad \frac{c\pi}{\Gamma(\nu)\sin(\nu - 1)\pi} (\rho\beta)^{\nu - 1} \\ &+ \quad \frac{c_1 \pi}{\Gamma(\nu_1)\sin(\nu_1 - 1)\pi} (\rho\beta)^{\nu_1 - 1}, \end{aligned} \tag{2.34}$$

with $g(\rho\beta)$ again a regular function for $\mathrm{Re}\, \rho > -\delta$ and which satisfies $g(0) = 0$ and (2.12). Proceeding with the analysis above with (2.11) replaced by (2.35) leads again to (2.20) since $\nu_1 > \nu$, cf. (1.1). The remaining part of the proof with $c_1 > 0$ does not differ from that with $c_1 = 0$, and so the theorem has been proved for $c_n > 0$ and ν_n not an integer.

Finally we have to consider the case $\nu_1 = k \geq 2$, with k an integer. We have, cf. [7], vol. I, p. 468,

$$c_1 \int_{\beta}^{\infty} \mathrm{e}^{-st} \frac{1}{(t/\beta)^{k}} \frac{\mathrm{d}t}{\beta} = -g_2(\rho\beta) + c_1 \frac{(-1)^k}{(k-1)!} (\rho\beta)^{k-1} \log(\rho\beta),$$

again with $g_2(\rho\beta)$ a regular function, and (2.35) becomes: for $\mathrm{Re}\, \rho \geq 0$,

$$1 - \frac{1 - \beta(\rho)}{\rho\beta} = g(\rho\beta) + \frac{c\pi(\rho\beta)^{\nu - 1}}{\Gamma(\nu)\sin(\nu - 1)\pi} + c_1 \frac{(-1)^{k-1}}{\Gamma(k-1)} (\rho\beta)^{k-1} \log(\rho\beta), \tag{2.35}$$

where $g(\rho\beta)$ is again an entire function for $\mathrm{Re}\, \rho > -\delta$, which satisfies $g(0) = 0$ and (2.12). The last term is $\mathrm{o}((\rho\beta)^{\nu - 1})$ since $k > \nu$. With this it is readily verified that the second equality sign in (2.20) also applies for the present case, and so the remaining part of the proof is similar to that with $c_1 = 0$. Hence the theorem has been proved.

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